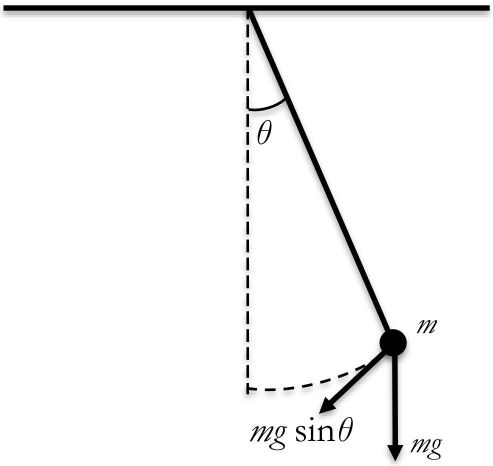
Deriving an expression for the time period of a simple pendulum

Consider a simple pendulum consisting of a light inextensible string of length l with a bob of mass m. The pendulum is pulled to one side and released. Prove that the resulting oscillations are simple harmonic and find an expression for the time period of the pendulum.



The restoring force (the force which constantly tried to return the pendulum to its equilibrium position) is given by:

$$F = -mg\sin\theta$$

From N(II), $F = ma$, so: $$ma = -mg\sin\theta$$

The ms cancel, giving: $$a = -g\sin\theta$$

We need to introduce displacement into this equation, since the defining equation for SHM is in terms of x. The instantaneous displacement of m will be the displacement measured along the arc length the pendulum travels through. Using this, we can say that $a = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}$ and $\theta = \frac{x}{l}$ (from the definition of a radian), so:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -g\sin\left(\frac{x}{l}\right)$$

It is not possible to solve this equation analytically for x*. However, making a reasonable approximation will sort this out. If we assume that θ is small (*i.e.* the pendulum is not swinging wildly backwards and forwards but only through small angles, as in a grandfather clock, for example) then:

$$\sin\theta \approx \theta = \frac{x}{l}$$

* A solution is possible, using elliptical functions and Maclaurin series to approximate: see footnote on next page.

So, our equation for *a* simplifies to: $a = -\frac{g}{l}x$ (for small amplitudes of oscillation†)

This clearly of the form $a = -\omega^2 x$, so a simple pendulum *does* undergo SHM for small amplitudes. Since $\omega = 2\pi f$ and $f = \frac{1}{T}$, it is now quite easy to show:

$$\boldsymbol{T = 2\pi\sqrt{\frac{l}{g}}}\ ‡$$

Another approach…

If we consider *m* as undergoing a twisting motion about the pivot, we can use slightly different mechanics to analyse the situation and see if we get the same result.

In linear dynamics, *F* = *ma*. In rotational dynamics, the equivalent equation is

$$M = I\alpha$$

M = Moment (= Force x distance from pivot),
I is the object's moment of inertia (a measure of how much it resists being turned). For a simple pendulum, $I = ml^2$.
α is the object's angular acceleration (*i.e.* $\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}$)

In the case of a pendulum, $M = F \times d = -mg \times l\sin\theta$. This gives:

$$-mgl\sin\theta = ml^2\alpha$$

Which simplifies to:

$$-\frac{g}{l}\sin\theta = \alpha$$

We now need to make exactly the same $\sin\theta \approx \theta$ approximation we made before to give:

$$-\frac{g}{l}\theta = \alpha$$

Which is of the form $a = -\omega^2 x$ (albeit in a rotational sense), so we can solve for T in the same way as before to give $\boldsymbol{T = 2\pi\sqrt{\frac{l}{g}}}$.

Thus both approaches give the same result (and require the same approximation), which is comforting to know.

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† 'Small' in this context means up to about 20 degrees. If this is so, the equation is valid to within 1%. As the oscillations get bigger, the pendulum will swing more slowly than this (*i.e.* *T* increases as angle of swing gets bigger).

‡ The solution which does **not** require the amplitude to be small is $T = 2\pi\sqrt{\frac{l}{g}}(1 + \frac{\theta^2}{16} + \frac{11\theta^2}{3072} + \frac{173\theta^2}{737280} + \cdots)$