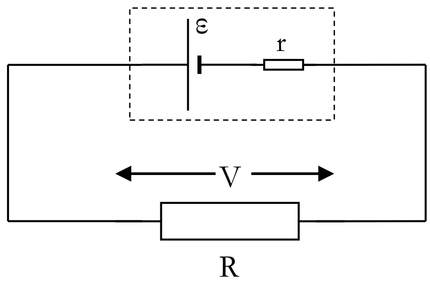
Maximum Power Theorem

Consider the following circuit:



What you should already know:
The emf generated by the supply (ε) is lost partially in the internal resistance (r) and partly in the Load resistance (R).

From Kirchoff II (or conservation of energy, if you prefer):

$$V = \varepsilon - Ir$$

How much power is delivered to R?
Since $I = \frac{V}{R}$, and total resistance $= (R + r)$, in this case $I = \frac{\varepsilon}{(R+r)}$.

Now $\quad P = VI = I^2R$

So, in this case: $\quad P = \left(\frac{\varepsilon}{(R+r)}\right)^2 R = \frac{\varepsilon^2 R}{(R+r)^2}$

Under what circumstances will maximum power be delivered to R?
To find a maximum, differentiate the expression for P and set the result equal to zero. Since the only variable which the user can directly vary is R, the differentiation must be done with respect to R:

$$P = \frac{\varepsilon^2 R}{(R+r)^2}$$

To differentiate a fraction, use the Quotient Rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$$

In this case, $u = \varepsilon^2 R$ so $\frac{\mathrm{d}u}{\mathrm{d}R} = \varepsilon^2$. Also, $v = (R+r)^2$ so $\frac{\mathrm{d}v}{\mathrm{d}R} = 2(R+r)$.

Substitute these values:

$$\frac{\mathrm{d}P}{\mathrm{d}R} = \frac{(R+r)^2\varepsilon^2 - (\varepsilon^2 R)(2(R+r))}{(R+r)^4}$$

In the numerator, take out a factor of $\varepsilon^2(R+r)$:

$$\frac{\mathrm{d}P}{\mathrm{d}R} = \frac{\varepsilon^2(R+r)[(R+r)-2R]}{(R+r)^4}$$

Cancel a factor of $(R+r)$ from top and bottom:

$$\frac{\mathrm{d}P}{\mathrm{d}R} = \frac{\varepsilon^2[(R+r)-2R]}{(R+r)^3}$$

This simplifies to:

$$\frac{\mathrm{d}P}{\mathrm{d}R} = \frac{\varepsilon^2(r-R)}{(R+r)^3}$$

The Power will be maximum when $\frac{\mathrm{d}P}{\mathrm{d}R} = 0$:

$$0 = \frac{\varepsilon^2(r-R)}{(R+r)^3}$$

$$0 = (r-R)$$

$$R = r$$

So, to get maximum power from the supply, you need the Load resistance (*R*) to equal the internal resistance (*r*). This is a simple example of what is called **impedance matching**, and is used when connecting amplifiers to loudspeakers: the resistance (actually the impedance, which is a complex equivalent of resistance for AC circuits) of the amplifier's output circuit must match that of the loudspeaker(s) if you want to deliver maximum power to the loudspeakers. A typical loudspeaker has an impedance of 8 Ohms, which is the impedance most amplifier output circuits are designed to have. Headphones tend to have much higher impedances, which is one reason why you can't connect headphones to loudspeaker outputs or vice versa and expect them to work.

Impedance matching isn't just an electrical phenomenon either: for example, ultrasound scans require the use of a water-based gel between the ultrasound probe and the body being probed. Without the gel, the air-gap between probe and body would reflect nearly all of the ultrasound back into the probe. The gel bridges the gap in such a way that the sound encounters a nearly constant (or relatively slowly-changing) acoustic impedance throughout its journey.

How efficient is the power supply when delivering maximum power?

You might think that all power supplies are designed to deliver maximum power, since this is what will always be desired. But it's not that simple:

$$\text{Efficiency} = \frac{\text{Power delivered to Load}}{\text{Power output of } \varepsilon}$$

$$\text{Efficiency} = \frac{VI}{\varepsilon I} = \frac{V}{\varepsilon}$$

Now, $V = IR$ and $\varepsilon = V + Ir$, so:

$$\text{Efficiency} = \frac{IR}{V + Ir}$$

Substitute $V = IR$ again:

$$\text{Efficiency} = \frac{IR}{IR + Ir}$$

At maximum Power, R = r, so:

$$\text{Efficiency} = \frac{IR}{IR + IR} = \frac{IR}{2IR} = \frac{1}{2}$$

So, if a device is set up to deliver maximum power, it <u>cannot</u> operate at more than 50% efficiency. This should make sense if you think about it: if R = *r* and *I* is the same through both resistances, the Power developed in each resistor must be the same (*i.e.* $I^2R = I^2r$) or, in other words, half the power from the supply is lost in each resistance.

This is why power amplifiers run hot: they are designed to output maximum power, but this means that half of the energy from the supply is inevitably lost in the internal resistance.

If R > r, then you don't operate at maximum power, but the efficiency can then improve beyond 50% (in fact, the efficiency asymptotically approaches 100% as R increases). This is how low-drain devices (*e.g.* mobile phones) are designed to work: you don't need huge power from the battery most of the time, but you do want it to be efficient so that the battery lasts longer.