Proving that Projectile Motion is Parabolic

A **projectile** is anything which moves two-dimensionally without any forces acting on it other than gravity. Real-life examples (if you ignore air resistance) include balls thrown through the air, water spurting from pipes and so on.

A **parabola** is a mathematical term for a curve which obeys the law:

$$y = A + Bx + Cx^2 + Dx^3 + \dots$$

(Where A, B, C etc. are constants)

The highest power of x in the equation tells you the order of the parabola (*i.e.* 1st order, 2nd order and so on). So, a graph of $y = x^2$ is a 2nd-order parabola.

Proof:

Consider an object of any mass, thrown at an initial angle θ to the ground, at an initial velocity v:

The components of v are:

Horizontally: $u_x = v\cos\theta$
Vertically: $u_y = v\sin\theta$

The components of the object's acceleration will be:

Horizontally: $a_x = 0$ (if we ignore air resistance)
Vertically: $a_y = -g$ (if we ignore air resistance)

At any time t into the object's flight:

$$v\cos\theta = \frac{x}{t}, \text{ so } t = \frac{x}{v\cos\theta}$$

Substitute this expression for t into $s = ut + \frac{1}{2}at^2$ and you can find the vertical displacement of the object at any time as a function of x (*i.e.* set $s = y$):

$$s = ut + \tfrac{1}{2}at^2$$

Substitute $s = y,\ u = v\sin\theta,\ t = \dfrac{x}{v\cos\theta}$ and $a = -g$:

$$y = (\cancel{v}\sin\theta)\left(\frac{x}{\cancel{v}\cos\theta}\right) + \frac{1}{2}(-g)\left(\frac{x}{v\cos\theta}\right)^2$$

Simplify ($\dfrac{\sin\theta}{\cos\theta} = \tan\theta$) and take common factors of x out of each term:

$$y = (\tan\theta)x - \left(\frac{g}{2v^2\cos^2\theta}\right)x^2$$

…which is now of the form:

$$y = Ax + Bx^2$$

q.e.d.

Calculating the range of a projectile

Range = value of x when $y = 0$, so:

$$0 = (\tan\theta)x - \left(\frac{g}{2v^2\cos^2\theta}\right)x^2$$

$$\cancel{x}\tan\theta = \frac{gx^{\cancel{2}}}{2v^2\cos^2\theta}$$

$$x = \frac{2v^2\tan\theta\cos^2\theta}{g}$$

$\tan\theta\cos^2\theta = \sin\theta\cos\theta$, so:

$$x = \frac{2v^2\sin\theta\cos\theta}{g}$$

$2\sin\theta\cos\theta = \sin 2\theta$, so:

$$\boxed{\text{Range} = x = \frac{v^2\sin 2\theta}{g}}$$

Range is maximum when $\sin 2\theta$ is maximum, *i.e.* $\sin 2\theta = 1$ so $2\theta = 90°$ or $\underline{\boldsymbol{\theta = 45°}}$