Proving that Projectile Motion is Parabolic

A **projectile** is anything which moves two-dimensionally without any forces acting on it other than gravity. Real-life examples (if you ignore air resistance) include balls thrown through the air, water spurting from pipes and so on.

A **parabola** is a mathematical term for a curve which obeys the law:

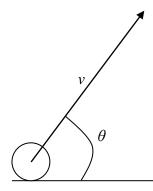
$$y = A + Bx + Cx^2 + Dx^3 + ...$$

(Where A, B, C etc. are constants)

The highest power of x in the equation tells you the order of the parabola (i.e. 1st order, 2^{nd} order and so on). So, a graph of $y = x^2$ is a 2^{nd} -order parabola.

Proof:

Consider an object of any mass, thrown at an initial angle θ to the ground, at an initial velocity v:



The components of v are:

Horizontally: $u_x = v \cos \theta$ Vertically: $u_y = v \sin \theta$

The components of the object's acceleration will be:

Horizontally: $a_x = 0$ (if we ignore air resistance) Vertically: $a_y = -g$ (if we ignore air resistance)

At any time *t* into the object's flight:

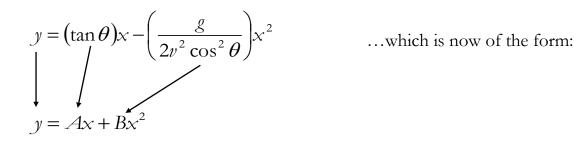
$$v\cos\theta = \frac{x}{t}$$
, so $t = \frac{x}{v\cos\theta}$

Substitute this expression for t into $s = ut + \frac{1}{2}at^2$ and you can find the vertical displacement of the object at any time as a function of x (*i.e.* set s = y):

$$s = ut + \frac{1}{2}at^2$$
 Substitute $s = y, u = v \sin \theta, t = \frac{x}{v \cos \theta}$ and $a = -g$:

$$y = (/ \sin \theta) \left(\frac{x}{/ \cos \theta} \right) + \frac{1}{2} \left(-g \left(\frac{x}{/ \cos \theta} \right)^2 \right)$$

Simplify $(\frac{\sin \theta}{\cos \theta} = \tan \theta)$ and take common factors of x out of each term:



q.e.d.

Calculating the range of a projectile

Range = value of x when y = 0, so:

$$0 = (\tan \theta)x - \left(\frac{g}{2v^2 \cos^2 \theta}\right)x^2$$

$$x \tan \theta = \frac{gx^2}{2v^2 \cos^2 \theta}$$

$$x = \frac{2v^2 \tan \theta \cos^2 \theta}{g}$$

 $\tan\theta\cos^2\theta = \sin\theta\cos\theta$, so:

$$x = \frac{2v^2 \sin \theta \cos \theta}{g}$$

 $2\sin\theta\cos\theta = \sin 2\theta$, so:

Range =
$$x = \frac{v^2 \sin 2\theta}{g}$$

Range is maximum when $\sin 2\theta$ is maximum, *i.e.* $\sin 2\theta = 1$ so $2\theta = 90^{\circ}$ or $\theta = 45^{\circ}$