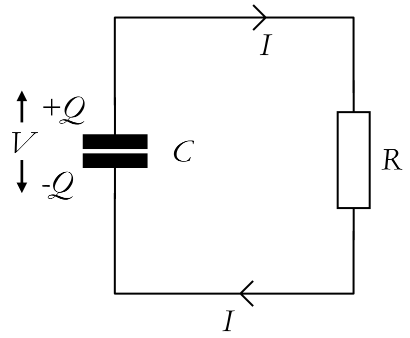
Proving that capacitor discharge is exponential

A capacitor of Capacitance C Farads has been charged with Q_0 Coulombs of charge and is discharging through a resistance R (which includes any resistance in the wires). At any moment, the p.d. across the capacitor is V:



Applying Kirchoff (II) around the circuit loop, in the direction of I:

$$V - IR = 0 \quad \text{so} \quad V = IR, \quad \text{or} \quad I = \frac{V}{R}$$

However, neither I nor V are constant; both are falling as the capacitor discharges. At any instant in time, $Q = CV$, so:

$$I = \frac{V}{R} = \frac{Q}{RC}$$

In any time δt, an amount of Charge equal to $I\,\delta t$ moves around the circuit. Hence the charge on the capacitor plates decreases by this amount during the time interval δt. The rate of change of charge on the plates is therefore:

$$-\frac{\mathrm{d}Q}{\mathrm{d}t} = I \quad \text{but} \quad I = \frac{Q}{RC} \quad \text{so} \quad \frac{\mathrm{d}Q}{\mathrm{d}t} = -\frac{Q}{RC}$$

To calculate the charge remaining on C at any time t, we need to solve this equation. This would also tell us the equation of the line on a Q-t graph (which we suspect to be exponential). To solve the equation, we need to separate the variables Q and t:

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = -\frac{Q}{RC} \quad \text{so} \quad \frac{\mathrm{d}Q}{Q} = -\frac{\mathrm{d}t}{RC}$$

and then integrate:

$$\int_{Q_0}^{Q} \frac{\mathrm{d}Q}{Q} = -\int_{0}^{t} \frac{\mathrm{d}t}{RC}$$

This might look slightly easier to cope with if we write it as:

$$\int_{Q_0}^{Q} \frac{1}{Q}\,\mathrm{d}Q = -\frac{1}{RC}\int_0^t 1 \ \mathrm{d}t$$

…and then integrate:

$$\therefore \quad [\ln Q]_{Q_0}^{Q} = -\frac{1}{RC}[t]_0^t$$

Substitute in the limits:

$$\ln Q - \ln Q_0 = -\left(\frac{t}{RC}\right) - \left(\frac{0}{RC}\right)$$

which simplifies to give:

$$\ln\frac{Q}{Q_0} = -\frac{t}{RC}$$

We want this equation in terms of Q (not ln Q), so we raise both sides to the power of e to remove the natural logarithms:

$$e^{\left(\ln\frac{Q}{Q_0}\right)} = e^{-t/RC}$$

Which, since $e^{\ln x} = x$, means:

$$\frac{Q}{Q_0} = e^{-t/RC}$$

Which rearranges onto one line as:

$$\boxed{Q = Q_0\ e^{-t/RC}}$$

Where:

Q = the charge on the capacitor at time t (in Coulombs),
Q_0 = the charge on the capacitor just before discharge started (in Coulombs),
t = time since discharge started (in seconds),
R = the Resistance the capacitor is discharging through (in Ohms),
and C = the capacitance of the capacitor (in Farads).

This is the equation of the line on a Q-t graph. The e^{-x} factor proves that the decay of charge on a capacitor is indeed exponential over time, as we suspected.

Time constant

The product of the discharge resistor's resistance and the capacitor's capacitance (RC) is called the **time constant** (= τ) of the capacitor circuit. This is the capacitor equivalent of a radioactive isotope's half-life. RC is the time taken for the charge on the capacitor to fall to a certain fraction (not ½, like in radioactive half-life) of its original value. To find out exactly what fraction it is, set $t = RC$ in the above equation, and see what happens…

The equation collapses to $Q = Q_0 e^{-1} = \frac{Q_0}{e}$, meaning:

> the time constant ($\tau = RC$) of a capacitor discharge circuit is the time it takes for the charge on the capacitor to fall to $\frac{1}{e}$ (≈ 37%) of its original value.

Discharging a Capacitor fully

In theory, because of the nature of an exponential function, a mathematician would say that a capacitor never fully discharges. Since Q asymptotically approaches zero over time, it can never exactly equal zero. However, the above proof assumed that charge was infinitely divisible (since we used an infinitesimally small change of Charge dQ) whereas, in practice, electrons are finite. A physicist therefore appreciates that, in practice, the final electron will eventually leave the negative plate, leaving the capacitor fully discharged.

Despite this, it is useful to have a rule of thumb to say at what time a capacitor is, to all intents and purposes, fully discharged. Calculate the percentage of the initial charge remaining on the capacitor after the following number of time constants:

Time after discharge starts	Percentage of initial charge remaining
0τ	100%
1τ	37%
2τ	
3τ	
4τ	
5τ	
6τ	

The usual rule of thumb is to say that **after 5τ the capacitor is** (to all intents and purposes) **fully discharged**, because then you have less than 1% of the initial charge remaining.